(2)

David Taylor Research Center
Bethesda, MD 20084–5000

DTRC–PAS–90–45 July 1990
Propulsion and Auxiliary Systems Department
Research and Development Report

A Differential Turbomachinery Equation with Viscous Correction

by

Herman B. Urbach

90

MAJOR DTRC TECHNICAL COMPONENTS

CODE 011	DIRECTOR OF TECHNOLOGY, PLANS AND ASSESSMENT
12	SHIP SYSTEMS INTEGRATION DEPARTMENT
14	SHIP ELECTROMAGNETIC SIGNATURES DEPARTMENT
15	SHIP HYDROMECHANICS DEPARTMENT
16	AVIATION DEPARTMENT
17	SHIP STRUCTURES AND PROTECTION DEPARTMENT
18	COMPUTATION, MATHEMATICS & LOGISTICS DEPARTMENT
19	SHIP ACOUSTICS DEPARTMENT
27	PROPULSION AND AUXILIARY SYSTEMS DEPARTMENT
28	SHIP MATERIALS ENGINEERING DEPARTMENT

DTRC ISSUES THREE TYPES OF REPORTS:

1. **DTRC reports, a formal series,** contain information of permanent technical value. They carry a consecutive numerical identification regardless of their classification or the originating department.

2. **Departmental reports, a semiformal series,** contain information of a preliminary, temporary, or proprietary nature or of limited interest or significance. They carry a departmental alphanumerical identification.

3. **Technical memoranda, an informal series,** contain technical documentation of limited use and interest. They are primarily working papers intended for internal use. They carry an identifying number which indicates their type and the numerical code of the originating department. Any distribution outside DTRC must be approved by the head of the originating department on a case-by-case basis.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION Unclassified	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution is unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S) DTRC–PAS–90–45	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION David Taylor Research Center — 6b. OFFICE SYMBOL (If applicable) Code 2720	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS (City, State, and ZIP Code) Annapolis, Maryland 21402–5067	7b. ADDRESS (City, State, and ZIP Code)
8a. NAME OF FUNDING/SPONSORING ORGANIZATION — 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c. ADDRESS (City, State, and ZIP Code)	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO. / PROJECT NO. / TASK NO. / WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification)

A Differential Turbomachinery Equation with Viscous Correction

12. PERSONAL AUTHOR(S)

Dr. Herman B. Urbach

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (YEAR, MONTH, DAY)	15. PAGE COUNT
Final	FROM 1989 TO 1990	1990 June 30	

16. SUPPLEMENTARY NOTATION Based upon a thesis submitted in partial fulfillment of the degree requirements of the Aerospace Department of the University of Maryland, College Park, Md. Professor Everett Jones, graduate thesis advisor.

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Differential turbomachinery equation; Viscous correction for turbomachinery equation

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

A differential turbomachinery equation describing the energy transfer between a fluid and any body moving in that fluid was derived. The derivation is based upon the Coriolis form of the Navier-Stokes Equations. A differential equation for the total relative rothalpy was also obtained. The equalities contain a rigorous viscous correction for the total enthalpy and rothalpy.

The differential equation defines the substantial derivative of the total enthalpy at any point in a fluid. This substantial derivative represents the energy transfer rate into or out of the fluid at the point. In order to obtain the total enthalpy transfer between the fluid and the body, it is necessary to integrate over the entire domain. For ideal flow regimes, the integration domain may be restricted entirely within the rotor. For real fluids, viscous coupling requires that regions outside the blade area be considered.

On integration of the differential equations, a form of the Euler Turbomachinery Equation with viscous correction is derived. The resultant form contains two distinct work rate terms for the axial and radial components of the flow. The fact that integration yields a result which approximates the classic Euler Turbomachinery Equation constitutes confirmation of the derivation.

An application of the equation to an ideal infinite linear cylinder with bound vorticity was developed. the cylinder was made to act like a turbine blade performing work by lifting an ideal airframe against gravity. The integration yielded the expected known result.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT	21. ABSTRACT SECURITY CLASSIFICATION
☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	Unclassified

22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE (Include Area Code)	22c. OFFICE SYMBOL
Dr. Herman B. Urbach	301–267–2864	Code 2720

DD FORM 1473, JUN 86 — Previous editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE

UNCLASSIFIED

0102–LF–014–6602

CONTENTS

FIGURES

NOMENCLATURE

$\bar{e}$	A unit vector
f	Any continuous function
h	Enthalpy per unit mass of fluid
h_o	Total enthalpy per unit mass of fluid
h_{ow}	Relative total specific rothalpy
H_o	Total enthalpy of a fluid; in linear systems, the total enthalpy of a fluid per unit length of blade
L	Lift per unit length of blade
m	Mass flow rate
n	A constant
p	Local static pressure
p_o	Total pressure
q	A generalized curvilinear coordinate
$\dot{q}$	A specific heat rate
r	Radial coordinate of a cylindrical coordinate system
$\bar{R}$	Position vector for the point of interest in a fluid
s	Entropy per unit mass of fluid
t	Time
T	Absolute temperature
$\bar{U}$	Velocity of the blade and a function of the radius
$\bar{V}$	Velocity
(VIS)	The integrated viscous term of Equation 27
$\bar{W}$	Relative fluid velocity in a moving rotor frame
z	Axial coordinate of a cylindrical coordinate system
δ_i^j	Kronecker's delta function
Γ	The circulation

ϕ	The dissipation
θ	Tangential angle coordinate in cylindrical coordinates
ν	The kinematic viscosity
$\bar{\pi}'$	The stress tensor separated from the thermodynamic pressure
ϱ	Local fluid density
ψ	A stream function
ω	Angular velocity

ABSTRACT

A differential turbomachinery equation describing the energy transfer between a fluid and any body moving in that fluid was derived. The derivation is based upon the Coriolis form of the Navier-Stokes Equations. A differential equation for the total relative rothalpy was also obtained. The equalities contain a rigorous viscous correction for the total enthalpy and rothalpy.

The differential equation defines the substantial derivative of the total enthalpy at any point in a fluid. This substantial derivative represents the energy transfer rate into or out of the fluid at the point. In order to obtain the total enthalpy transfer between the fluid and the body, it is necessary to integrate over the entire domain. For ideal flow regimes, the integration domain may be restricted entirely within the rotor. For real fluids, viscous coupling requires that regions outside the blade area be considered.

On integration of the differential equations, a form of the Euler Turbomachinery Equation with viscous correction is derived. The resultant form contains two distinct work rate terms for the axial and radial components of the flow. The fact that integration yields a result which approximates the classic Euler Turbomachinery Equation constitutes confirmation of the derivation.

An application of the equation to an ideal infinite linear cylinder with bound vorticity was developed. The cylinder was made to act like a turbine blade performing work by lifting an ideal airframe against gravity. The integration yielded the expected known result.

ADMINISTRATIVE INFORMATION

This study was performed in partial fulfillment of thesis requirements of the Aerospace Department of the University of Maryland at College Park. Professor Everett Jones was the graduate advisor and director. Publication of this manuscript was supported by the David Taylor Research Center under the Independent Research and Development Program (IR&IED) under Dr. Bruce Douglas.

The wok has significant implications to turbomachinery studies of efficiency and noise in the Power Systems Division, Code 272, of the Propulsion and Auxiliary Systems Department, which has supported this publication.

ACKNOWLEDGMENT

The author is thankful to his supervisors, Messrs. Timothy Doyle and John R. Ward for their enouragement and confidence and Drs. Earl Quandt and Bao Hwang for useful discussions.

INTRODUCTION

A monument of turbomachinery technology is Euler's Turbomachinery Equation which is based upon thermodynamic definitions of work and Newton's Laws. Since the Navier-Stokes Equations and Crocco's Equation[1,2] in a rotating (moving) frame are also based upon thermodynamics and Newton's Laws, they must in principle contain Euler's Turbomachinery Equation in differential form and, on integration, in integral form.

Integration of Crocco's Equation in the absolute frame has given rise to the "Unsteadiness Paradox" to explain rotor energy transfer.[3–14] An aspect of the paradox which must be rationalized (vide infra) is Vavra's statement (ref. 8, pp. 110 and 111) that if the relative velocity and the rotor velocity are constant with time, then the absolute velocity must be time independent.

Time-dependence is largely ignored in the design of marine propellers. The fact that energy transfer is routinely calculated in a steady-state moving frame provides philosophical questions concerning the proper interpretations of the Unsteadiness Paradox. This rationalization is effected quite simply by the "crypto-steady" relation that arises from the Galilean transformation used by Coriolis.

A Galilean transformation with rotation (the Coriolis transformation) that connects the moving rotor frame and the absolute or laboratory frame provides a relationship between the frames so that integration of the energy rate may be conveniently performed in a time-independent frame with a time-independent set of coordinates.

The transformation leads to simplified expressions for the substantial total-enthalpy transfer rate which are uncoupled from the expressions for the substantial entropic energy rate. The fact that the total enthalpy and the entropic energy rates are uncoupled makes for simplified integration of the total enthalpy transfer, and generates a differential form of the turbomachinery equation and, on integration, a novel form of Euler's Turbomachinery Equation corrected for viscous non-ideal flow.

Euler's classical Turbomachinery Equation does not include a viscous term to correct for viscous losses since Euler was concerned only with the mechanical work transfer to or from the shaft. If one is concerned with total enthalpy transfer at a point in a fluid, *the viscous loss must be addressed* as is illustrated in this paper.

An application of the differential turbomachinery equation is described for a two-dimensional, ideal, linear turbine.

THE GALILEAN TRANSFORMATION

The moving frame and the absolute frame of a turborotor are connected by a Galilean transformation, which imposes relationships between the coordinates of the frames. From these relationships the vector operations in the two frames may be derived.

In the following discussion the subscripts v and w represent the absolute and the moving frame coordinate and vector values. (See Fig. 1 and the Nomenclature for definitions of quantities.)

Following Spannhake,[3] derivatives in the moving coordinate and time have the following relationship:

$$\frac{1}{r}\left(\frac{\partial}{\partial \theta_w}\right)_{\theta_v} = -\frac{1}{r\omega}\left(\frac{\partial}{\partial t}\right)_{\theta_v} = -\frac{1}{U}\left(\frac{\partial}{\partial t}\right)_{\theta_v} . \qquad (1)$$

Similar results are obtained with cartesian coordinates.

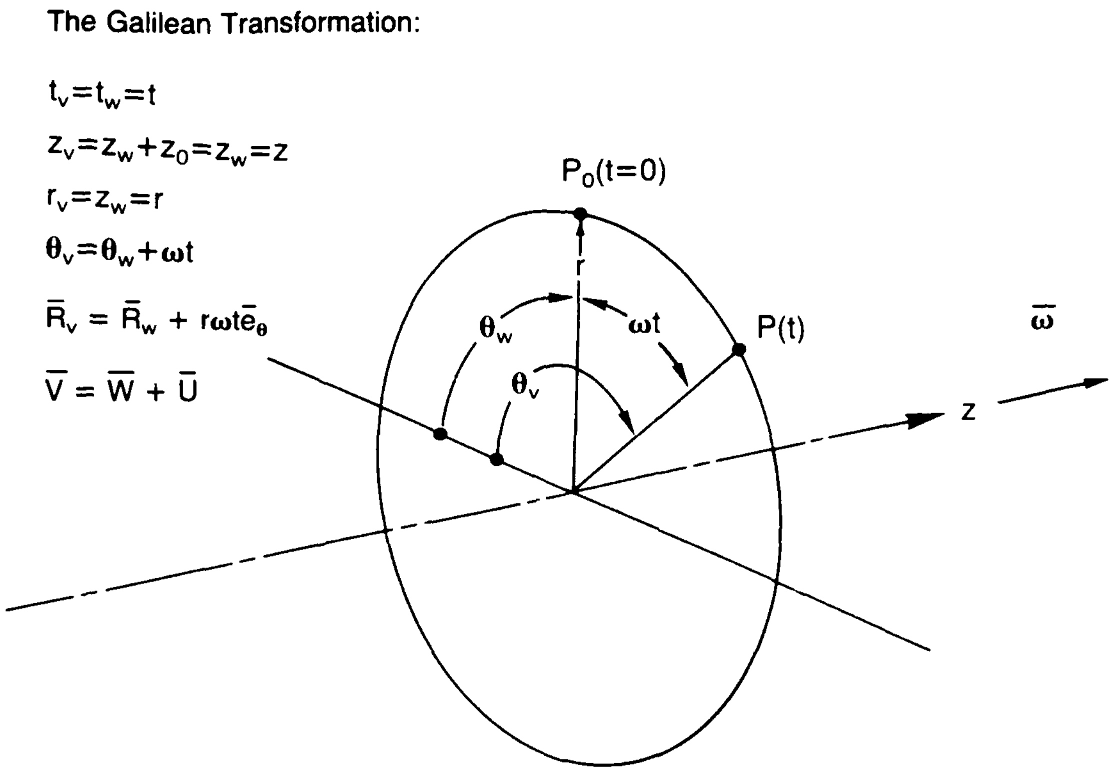


Fig. 1. Configurational relationships between the absolute and moving coordinate systems.

$$\left(\frac{\partial}{\partial y_w}\right)_{y_v} = -\frac{1}{U}\left(\frac{\partial}{\partial t}\right)_{y_v} . \tag{2}$$

Equations 1 and 2 define, in fact, the "crypto-steady criterion." [15] If U is constant, then a frame exists in which the flow regime is truly steady state.

The vector operator ∇ is independent of time. Therefore

$$\nabla_v = \nabla_w = \nabla . \tag{3}$$

Now from Eq. 1 for any function f,

$$\left(\frac{\partial f}{\partial t}\right)_{q_i} = -U\left(\frac{\partial f}{\partial q_{wu}}\right) = -\overline{U} \cdot \nabla_w f = -\overline{U} \cdot \nabla f . \tag{4}$$

where q_i represents all the position coordinates in the absolute frame and q_{wu} is the moving-frame coordinate of a generalized curvilinear orthogonal coordinate system which completely defines the position and velocity of the energy-transferring device. Eq. 4 is an extension of the usual crypto-steady relation.

THE INTERPRETATION OF TIME DERIVATIVES

The time derivative of the static blade-to-blade pressure is obtained from Eq. 4 by substituting p for the function f. Using cylindrical coordinates

$$\left(\frac{\partial p}{\partial t}\right)_{r,\theta_v,z} = -\frac{U}{r}\left(\frac{\partial p}{\partial \theta_w}\right)_{r,t,z} = -\frac{\partial p}{\partial(\theta_w/\omega)_t} \ . \tag{5}$$

Figure 2a shows a point $P(r,z,t)$ fixed in the absolute frame between blades of a centrifugal compressor rotor rotating into decreasing values of θ_v. The thermodynamic properties of the point P at the suction side of blade 1 change as time advances and the pressure side of blade 2 approaches P. When the wall passes through point P, fluid properties cease to exist at P. Thermodynamic information about the fluid at this point ceases. The time interval Δt for n blades and angular velocity ω is

$$t_2 - t_1 = \Delta t(r,z) = [2\pi/n - \delta(r,z)]/\omega \ , \tag{6}$$

where $\delta(r,z)$ is the subtended blade angle in radians.

Time is fixed at t_0 for the spatial derivatives of Fig. 2b. The angular change $\Delta\theta_w$ is in a direction negative to the blade motion.

Figure 2c shows a curve of the intra-blade pressure distribution changing with time in the absolute frame. In the moving frame, it represents the spatial distribution in a fixed instant of time.

TIME DEPENDENCE AND FRAME OF REFERENCE

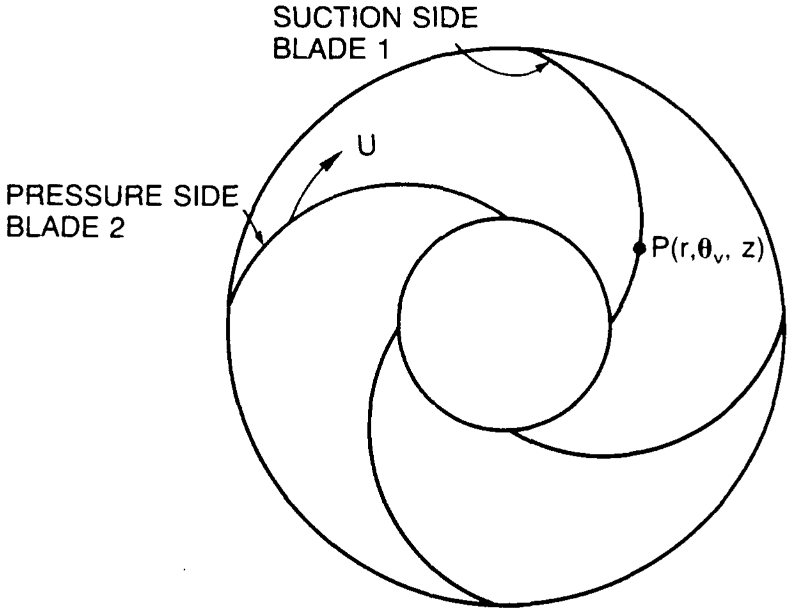
Looking at the relationship between moving and absolute frame velocities, namely

$$\overline{V} = \overline{W} + \overline{U} \ , \tag{7}$$

Vavra has noted that Eq. 7 suggests that if $\overline{W}$ is independent of time, then $\overline{V}$ is also independent of time. This point is an overlooked aspect of the Unsteadiness Paradox.[3–14] The converse statement is also true, and the Unsteadiness Paradox would imply that flow must be unsteady in the moving frame. The observer who sits on an ideal rotor in an ideal infinite fluid sees no time dependence in measured thermodynamic properties at a point. However, when the same observer passes to the absolute frame, he measures time-dependent thermodynamic properties each time a blade passes by that point.

The problem is resolved by noting that the frame of the coordinates used in Eq. 7 determines whether the observer perceives time dependence. If the velocity vectors are expressed in terms of r, θ_w, and z, the measurements are in the moving frame and both $\overline{V}(r,\theta_w,z)$ and $\overline{W}(r,\theta_w,z)$ are time independent. Writing $\overline{V}$ and $\overline{W}$ in the absolute

a) Absolute frame
P(r, θ_v, z) is fixed in space. Time changes. Blades rotate.



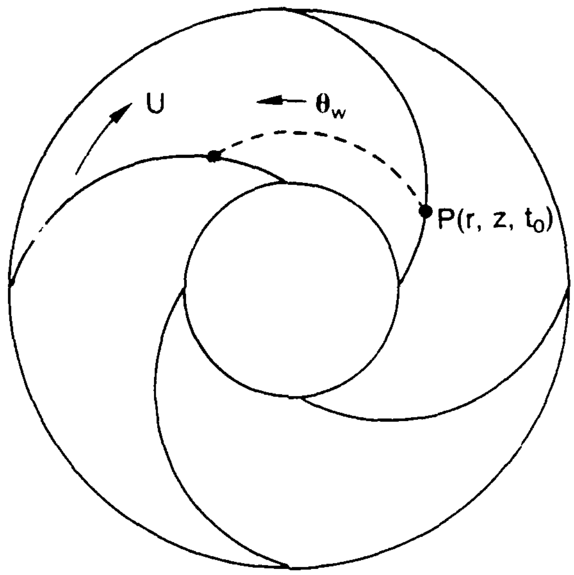
$$\left(\frac{\partial p}{\partial t}\right)_{r,\theta_v, z} = \frac{-u}{r}\left(\frac{\partial p}{\partial \theta_w}\right)_{r, z, t}$$

OVER TIME, Δt, THE PRESSURE SIDE OF THE NEXT BLADE MOVES TO THE POINT P(r, θ_v, z).

Fig. 2. Interpretation of the time-dependent pressure term in rotor flow: (a) absolute frame, (b) moving frame, and (c) graphical representation of the pressure over time and angle θ_w.

b) Moving frame

$P(r, z, t_0)$ over a range of decreasing θ_w in a fixed instant, t_0. Blades appear stationary.



c) Graphical representation of the pressure over time and angle θ_w.

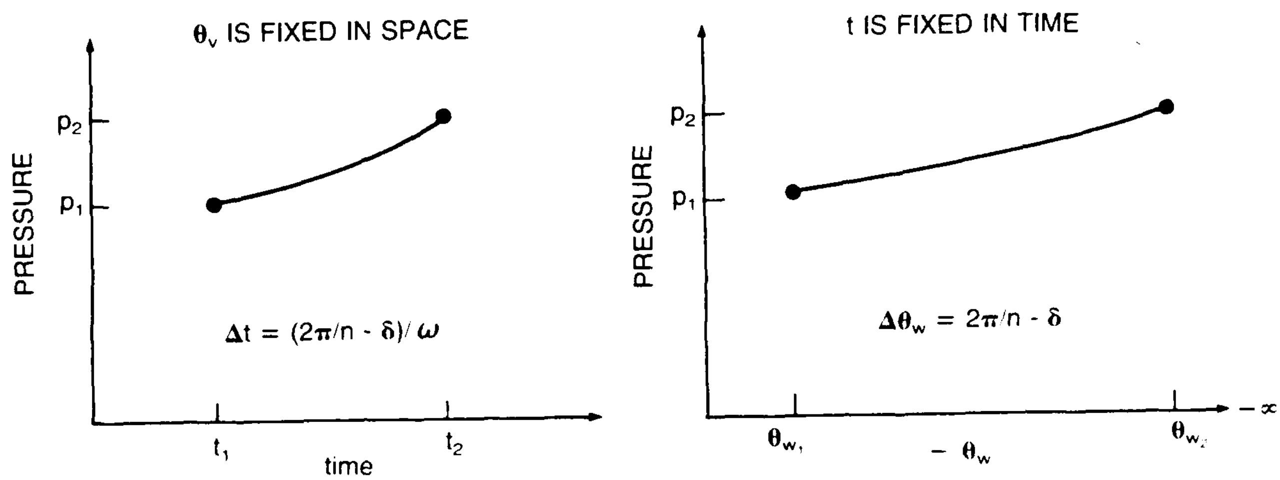


Fig. 2. Interpretation of the time-dependent pressure term in rotor flow (cont.)

coordinates requires that θ_w be written $\theta_v - \omega t$. Therefore, $\overline{V}(r, \theta_v - \omega t, z)$ and $\overline{W}(r, \theta_v - \omega t, z)$ are both time dependent in the absolute frame.

DERIVATION OF THE DIFFERENTIAL AND INTEGRAL FORMS

ASPECTS OF TIME-DEPENDENCE IN THE ABSOLUTE AND MOVING FRAMES

The Navier-Stokes Equations exhibit Coriolis terms in a unique steady-state rotating frame which develops from the Galilean transformation. Even in the frames of a device without rotation, the definition of the moving frame inherently contains and always invokes the crypto-steady relation, Eqs. 4 and 5. Provided there is uniform upstream flow, an observer moving with the device (for example a sail) may observe crypto-steady-state behavior in the fluid. Nevertheless, the time-dependent terms must be recovered on imposition of the Galilean transformation to the absolute frame.

In the Coriolis form the Navier-Stokes Equations[8] explicitly exhibit the rotational motion of the energy-transferring rotor device, thus:

$$\frac{\partial \overline{V}}{\partial t} + \overline{V} \cdot \nabla \overline{V} = \frac{\partial \overline{W}}{\partial t} + \overline{W} \cdot \nabla \overline{W} + 2\overline{\omega} \times \overline{W} - \nabla U^2/2$$

$$= -\frac{\nabla p}{\varrho} + \frac{1}{\varrho} \nabla \cdot \overline{\overline{\pi}}' \qquad (8)$$

where $\overline{\overline{\pi}}'$ represents the stress tensor excluding the pressure term $p\delta_i^j$.

THE COUPLED SUBSTANTIAL TOTAL ENTHALPY AND ENTROPIC ENERGY RATE IN THE MOVING AND ABSOLUTE FRAMES

If crypto-steady flow does not characterize the flow regime of the moving frame, the thermodynamic properties of the flow must fluctuate about an average value for any given thermodynamic condition. If $\overline{U}$ is constant, the time-dependent, linear acceleration term evaluated in the relative frame integrated over time cannot lead to energy transfer. However, time averages over non-linear terms lead to non-vanishing Reynold's stress terms. It will be assumed that time dependence in the moving frame is negligible. Therefore in Eq. 8 the acceleration term in the moving frame will be dropped.

$$\frac{\partial \overline{V}}{\partial t} + \nabla V^2/2 - \overline{V} \times (\nabla \times \overline{V})$$

$$= \nabla W^2/2 - \overline{W} \times (\nabla \times \overline{W}) + 2\overline{\omega} \times \overline{W} - \nabla U^2/2$$

$$= \frac{-\nabla p}{\varrho} + \frac{1}{\varrho} \nabla \cdot \overline{\overline{\pi}}' \ . \qquad (9)$$

Using the absolute-frame equality of Eq. 9 and the gradient form of Gibb's equation of state, the substantial derivative of the total enthalpy is obtained in terms of the partial

derivative of the pressure, the substantial entropic energy, $T\frac{Ds}{Dt}$, and the stress tensor $\bar{\bar{\pi}}'$ (see refs. 16–18 for details.)

$$\frac{Dh_o}{Dt} = \frac{1}{\varrho}\left(\frac{\partial p}{\partial t}\right)_{x_i} + T\frac{Ds}{Dt} + \frac{\bar{V}}{\varrho} \cdot (\nabla \cdot \bar{\bar{\pi}}') \ . \tag{10}$$

If the substantial entropic energy is independent of heat transfer, then it may be expressed as functions of the velocity vector along with the stress term. At this point analysis stops unless the partial pressure derivative is resolved in terms of the velocity. Employing Eq. 4 with the static pressure as the arbitrary function the result is

$$\frac{Dh_o}{Dt} = -\frac{\bar{U}}{\varrho} \cdot \nabla p + T\frac{Ds}{Dt} + \frac{\bar{V}}{\varrho} \cdot (\nabla \cdot \bar{\bar{\pi}}') \ . \tag{11}$$

where Eq. 9 may be used to replace the pressure derivative in Eq. 11. From the dot product of $\bar{U}$ with the moving frame equality in Eq. 9 the pressure gradient term becomes

$$\frac{-U}{\varrho r}\frac{\partial p}{\partial \theta_w} = -\bar{U} \cdot \frac{\nabla p}{\varrho} = \bar{U} \cdot (\bar{W} \cdot \nabla)\bar{W} + 2\bar{U} \cdot \bar{\omega} \times \bar{W}$$

$$- \bar{U} \cdot \nabla U^2/2 - \frac{1}{\varrho}\bar{U} \cdot (\nabla \cdot \bar{\bar{\pi}}') \ . \tag{12}$$

In Eq. 12 the gradient of the pressure is of course invariant in all frames, but the moving frame is convenient and preferable. Combining Eq. 11 with 12, the substantial derivative of the total enthalpy coupled with the entropic energy is obtained in terms of flow field variables.

$$\frac{Dh_o}{Dt} - T\frac{Ds}{Dt} = \bar{U} \cdot (\bar{W} \cdot \nabla)\bar{W} + 2\bar{U} \cdot \bar{\omega} \times \bar{W} + \frac{1}{\varrho}\bar{W} \cdot (\nabla \cdot \bar{\bar{\pi}}') \ . \tag{13}$$

UNCOUPLING THE SUBSTANTIAL DERIVATIVE OF THE TOTAL ROTHALPY AND ENTROPIC ENERGY RATES

If the gradient of the specific kinetic energy in the absolute frame is subtracted from the right-hand equality in Eq. 9 and $\bar{V}$ is replaced by Eq. 7, then[18]

$$-\nabla W_u U - \nabla U^2 + (\nabla \times \bar{W} + 2\bar{\omega}) \times \bar{W} = -\nabla h_o + T\nabla s + \frac{1}{\varrho}\nabla \cdot \bar{\bar{\pi}}' \ . \tag{14}$$

A useful intermediate expression is

$$h_{ow} \equiv h + \frac{V^2}{2} - UW_u - U^2 = h + \frac{W^2}{2} - \frac{U^2}{2} \ , \tag{15}$$

where the quantity h_{ow} in Eq. 15 is the total relative rothalpy. Combining Eqs. 14 and 15, Crocco's Equation for the total relative rothalpy in the moving frame with crypto-steady flow is derived.

$$\nabla h_{ow} - T\nabla s = \overline{W} \times (\nabla \times \overline{W} + 2\overline{\omega}) + \frac{1}{\varrho}\nabla\overline{\overline{\pi}}' \ . \tag{16}$$

Now, since the upstream flow is thoroughly mixed and without energy gradients, time derivatives of the pressure, rothalpy and entropy vanish in the moving frame. Thus

$$\frac{Dh_{ow}}{Dt} = \frac{\partial h_{ow}}{\partial t} + \overline{W} \cdot \nabla h_{ow} = \overline{W} \cdot \nabla h_{ow} \ , \tag{17}$$

and

$$T\left(\frac{Ds}{Dt}\right)_w = T\frac{\partial s}{\partial t} + T\overline{W} \cdot \nabla s = T\overline{W} \cdot \nabla s = T\left(\frac{Ds}{Dt}\right)_v \ . \tag{18}$$

where the last equality in Eq. 18 indicates that the entropic energy rate is invariant in all frames. Taking the dot product of $\overline{W}$ on Eq. 16 yields

$$\frac{Dh_{ow}}{Dt} = T\frac{Ds}{Dt} + \frac{1}{\varrho}\overline{W} \cdot (\nabla \cdot \overline{\overline{\pi}}') \ . \tag{19}$$

Now Wu notes that in the moving frame following a particle of fluid (ref. 17, p. 91)

$$T\frac{Ds}{Dt} = \dot{q} + \frac{1}{\varrho}(\overline{\overline{\pi}}' \cdot \nabla) \cdot \overline{W} \tag{20}$$

where $\dot{q}$ arises from heat transfer and the second term is the specific dissipation. It is assumed that there are no external heat sources or sinks and no thermal conduction and that crypto-steady flow prevails. Therefore

$$T\frac{Ds}{Dt} = \frac{1}{\varrho}(\overline{\overline{\pi}}' \cdot \nabla) \cdot \overline{W} \equiv \frac{1}{\varrho}\phi \ , \tag{21}$$

where ϕ is the dissipation. Combining Eqs. 19 through 21, the desired result is

$$\frac{Dh_{ow}}{Dt} = \frac{1}{\varrho}\nabla \cdot (\overline{\overline{\pi}}' \cdot \overline{W}) \ . \tag{22}$$

Equation 22 asserts that the substantial derivative of the total relative rothalpy is dependent upon the viscous dissipation. Using Eqs. 17 and 22 for inviscid flows without heat sources or sinks, and with crypto-steady characteristics

$$\nabla h_{ow} = \nabla\left(h + \frac{W^2}{2} - \frac{U^2}{2}\right) = \nabla(h_o - UV_u) = 0 \ . \tag{23}$$

Equation 23 represents differential or gradient forms of Euler's Turbomachinery Equations in the moving and absolute frame. Integration over a stream tube in the absolute frame yields the classic Euler Equation.

$$\Delta h_o = \Delta(UV_u) \ . \tag{24}$$

Note that Eq. 24 is strictly true only for isentropic flow. The substantial total enthalpy rate in the absolute frame is now obtained by eliminating the entropic energy rate in Eq. 13 with Eq. 21.

$$\frac{Dh_o}{Dt} = \overline{U} \cdot (\overline{W} \cdot \nabla)\overline{W} + 2\overline{U} \cdot \overline{\omega} \times \overline{W} + \frac{1}{\varrho}\nabla \cdot (\overline{\overline{\pi}}' \cdot \overline{W}) \ . \tag{25}$$

Equation 25, expressing the power transfer, is a differential form of the turbomachinery equation in terms of the moving frame.

With Eq. 21 it is now possible to uncouple the total-enthalpy, pressure relationship of (11) from the entropic energy.

$$\frac{Dh_o}{Dt} = -\frac{\overline{U}}{\varrho} \cdot \nabla p + \frac{1}{\varrho}\nabla \cdot (\overline{\overline{\pi}}' \cdot \overline{W}) + \frac{1}{\varrho}\overline{U} \cdot (\nabla \cdot \overline{\overline{\pi}}') \ . \tag{26}$$

Equation 26 indicates that neither axial nor radial pressure gradients are germane to the calculation of specific total enthalpy transfer. (Mass flow rates are of course a function of axial or radial pressure gradients.) Note, in contrast with some views (ref 9, pp. 7 and 8), only transverse pressure gradients parallel to $\overline{U}$ contribute to total enthalpy transfer. The impulse stages of turbomachines prove that axial or radial pressure gradients play no role in energy transfer. Moreover, from Eq. 25 for ideal flows which have no vorticity, only kinetic energy gradients paralleling the blade motion $\overline{U}$ contribute to total enthalpy transfer. Nevertheless, non-ideal total enthalpy transfer depends upon non-linear as well as the linear properties of the flow.

The substantial total enthalpy rate given by Eq. 25 is a differential form of the turbomachinery equation in the time-independent coordinates of the moving frame. A proper test of Eq. 25 would be the applicability of the equation to integration over the rotor blade-to-blade flow. Moving-frame integration should predict a total enthalpy transfer compatible with that of the Euler Turbomachinery Equation. Therefore, the integration of Eq. 25 will be performed as a test in the three-dimensional domain. An application of the new equation will be developed in a two-dimensional linear turbine. Compatibility of the results with Euler's Equation or the consequences thereof will lend credence to the logic of analysis employed in the derivation.

INTEGRATION OF THE TOTAL ENTHALPY RATE

In the integration process it will be assumed that the flow may be divided into streams which pass between a given pair of blades. In the rotating frame the streamtube walls are fixed steady-stage walls associated with a steady-state mass flow rate m which may consist of radial and axial mass flow components.

DERIVATION OF THE INTEGRAL FORM FROM THE DIFFERENTIAL FORM

The differential form of the turbomachinery equation (Eq. 25) may be integrated to yield

$$\int\int\int \varrho \frac{Dh_o}{Dt} d\tau$$

$$= \int\int\int \varrho \left[\overline{U} \cdot (\overline{W} \cdot \nabla)\overline{W} + 2\overline{U} \cdot \overline{\omega} \times \overline{W}\right] rdrd\theta dz$$

$$+ \int\int\int \nabla \cdot (\bar{\bar{\pi}}' \cdot \overline{W})\ rdrd\theta dz\ . \tag{27}$$

The first term of the right member of Eq. 27 is the tangential component of the convective term obtained on dot multiplication with $\overline{U}$, i.e.:

$$\varrho\overline{U} \cdot (\overline{W} \cdot \nabla)\overline{W}$$

$$= \varrho U\left(W_r \frac{\partial W_\theta}{\partial r} + \frac{W_\theta}{r}\ \frac{\partial W_\theta}{\partial \theta} + W_z \frac{\partial W_\theta}{\partial z} + \frac{W_r W_\theta}{r} \right)\ . \tag{28}$$

The first and last terms of Eq. 28 will be combined in an integral identified by $I_{1,4}$ thus:

$$I_{1,4} = \int\int\int \varrho r\omega \frac{W_r}{r}\ \frac{\partial r W_\theta}{\partial r} rdrd\theta dz\ . \tag{29}$$

The factors in Eq. 29 may be rearranged to express the radial mass flow which must be constant in steady flow.

Following arguments presented previously[18]

$$I_{1,4} = m_r \int d(U\bar{\bar{W}}_\theta) = m_r[(U\bar{\bar{W}}_\theta)_{r2} - (U\bar{\bar{W}}_\theta)_{r1}]\ , \tag{30}$$

where

$$\bar{\bar{W}}_\theta(r) = \frac{1}{\Delta\theta\Delta z} \int \int_{\theta_{min}}^{\theta_{max}} W_\theta\ d\theta dz\ . \tag{31}$$

The third term of the right member of Eq. 28 may be written to show the axial mass flow rate m_z explicitly.

$$I_3 = \int \left(\int\int \varrho \bar{\bar{V}}_z rdrd\theta \right) \frac{\partial \bar{U}\bar{\bar{W}}_\theta}{\partial z} dz\ , \tag{32}$$

where the axial velocity $\bar{\bar{V}}_z$, $\bar{\bar{W}}_\theta$, and $\bar{U}$ have been averaged over Δr and $\Delta\theta$. Substituting the axial mass rate for the parenthesis in Eq. 32,

$$I_3 = m_z \int d(\bar{U}\bar{\bar{W}}_\theta) = m_z[(\bar{U}\bar{\bar{W}}_\theta)_{z2} - (\bar{U}\bar{\bar{W}})_{z1}]\ . \tag{33}$$

The second term of the right member provides an integral which contains the tangential kinetic energy.

$$I_2 = \int\int \frac{\varrho\omega r}{2}\left(\int \frac{\partial W_\theta^2}{\partial\theta}d\theta\right)drdz$$

$$= \int\int \frac{\varrho\omega r}{2}\left[W_\theta^2(\theta_2) - W_\theta^2(\theta_1)\right]drdz = 0 \ . \tag{34}$$

Since the tangential velocities at the blade walls are equal to the blade velocity, the integral vanishes.

Now identifying the second term of the right member of Eq. 27 as I_5, we may write

$$I_5 = \int\int\int 2\varrho\omega^2 r W_r r dr d\theta dz \ . \tag{35}$$

Finally, following the arguments above,

$$I_5 \approx m_r\left[(U^2)_{r2} - (U^2)_{r1}\right] \ . \tag{36}$$

Summing the components of integration $I_{1,4}$ through I_5,

$$\int\int\int \varrho\frac{Dh_o}{Dt}d\tau = m_r\left[\Delta_r(U\bar{\bar{W}}_\theta) + \Delta_r(U^2)\right]$$

$$+ m_z\Delta_z(\tilde{U}\bar{\bar{W}}_\theta) + (VIS) \tag{37}$$

where Δ_r and Δ_z represent the change along r and z respectively, and (VIS) is the integrated viscous term.

Now adding $m_z\Delta_z\tilde{U}^2$, which is zero, to Fq. 37,

$$\int\int\int \varrho\frac{Dh_o}{Dt}d\tau = m_r\Delta_r\left[(\tilde{U} + \bar{\bar{W}}_\theta)\tilde{U}\right]$$

$$+ m_z\Delta_z\left[(U + \bar{\bar{W}}_\theta)U\right] + (VIS) \ . \tag{38}$$

In Eq. 38 the terms $\bar{\bar{W}}_\theta$ are averaged over θ and z in the first term and over r and θ in the second term. If the total steady-state mass rate m between a pair of blades is

$$m = m_r + m_z \ , \tag{39}$$

then,

$$\Delta h_o = f_r \Delta_r(\bar{U}\bar{\bar{V}}_\theta) + f_z \Delta_z(\bar{U}\bar{\bar{V}}_\theta) + (VIS)/m \ . \tag{40}$$

Equation 40 represents Euler's Turbomachinery Equation with mixed flows and the terms $\bar{\bar{V}}_\theta$ and $\bar{U}$ are averaged over the blade space where necessary. The coefficients f_r and f_z represent the radial and axial fractions of the mass flow.

The viscous term is a novel feature of the derivation which may explain losses of rotors during windmilling. With the exception of the viscous energy term *(VIS)*, the integral expression Eq. 40 exhibits a formal similarity and compatibility with Euler's Turbomachinery Equation. The derivation lends credence to the hypothesis that Eq. 25 is indeed a differential form of the turbomachinery equation (Eq. 24).

A two-dimensional application and test of the differential form (Eq. 25) on an ideal linear device where the solution is known precisely will now be examined.

THE SUBSTANTIAL TOTAL ENTHALPY RATE IN A TWO-DIMENSIONAL DEVICE

An infinite circular cylinder with bound circulation, as shown in Fig. 3, is an elemental linear turbine. It may be considered as an infinite sail on a sailboat or an infinite wing on a sailplane. The device extracts energy from the ideal inviscid working fluid. Work is performed on the sailplane (fixed to a vertical rail) by raising its height at uniform speed U against gravity. Work on the sailboat is performed by moving the boat at uniform speed U which elevates a weight attached at minus infinity by an infinite tether. In the moving frame the apparent velocity of the ideal working fluid at infinite distance is W_o. The relationship between the absolute and moving coordinate system and the velocities is given by the Galilean transformation of Fig. 1.

THE TOTAL ENTHALPY TRANSFER RATE BASED ON AERODYNAMICS

In the absence of vorticity and viscosity the ideal flow field will exhibit no total gradient in accordance with Eq. 23. Since the flow field is ideal, the flow domain may be described by a potential function or its conjugate stream function. The lift is therefore the ideal lifting force, L, of the Kutta-Joukowski Equation given by

$$L = \varrho W_o \Gamma \ , \tag{41}$$

where Γ is the scalar circulation. The units are force per unit length of cylinder. In the absolute and moving frame the lift component L_y directed parallel to the y axis of Fig. 3 is given by

$$L_y = \varrho W_{ox} \Gamma = \varrho V_{ox} \Gamma \ , \tag{42}$$

where the subscript x represents the x component. Recalling that U is the velocity of motion of the device (sail or wing or rotating cylinder) as perceived in the absolute frame, the power is the product of U and L_y.

$$Power/unit\ length = \varrho U W_{ox} \Gamma \ . \tag{43}$$

Since we assume that there is no heat rate,

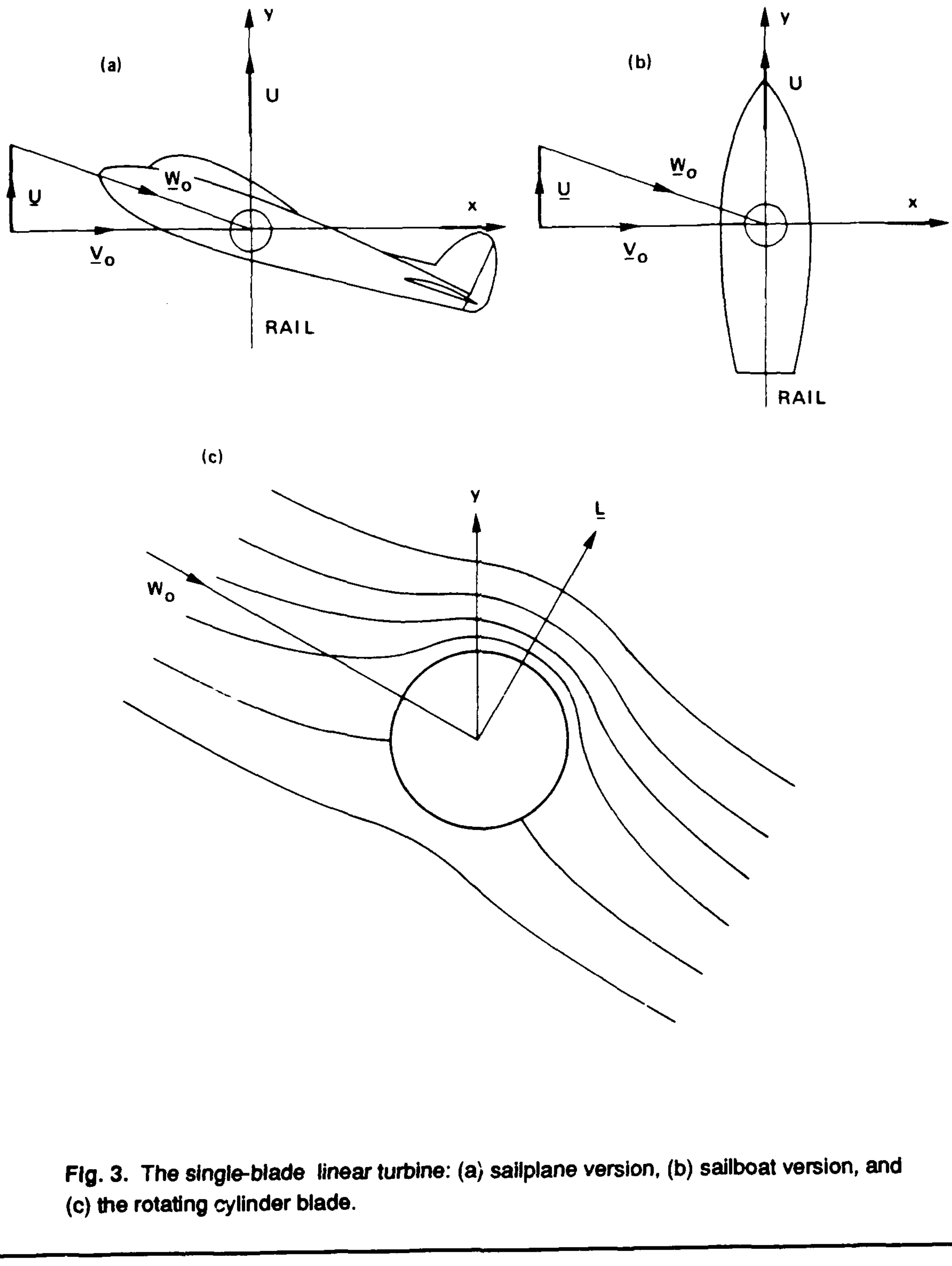


Fig. 3. The single-blade linear turbine: (a) sailplane version, (b) sailboat version, and (c) the rotating cylinder blade.

$$\frac{DH_o}{Dt} = -\varrho U W_{ox} \Gamma \ , \tag{44}$$

where H_o is the total enthalpy of the system per unit length. Equation 44 is the anticipated relationship which should ultimately be developed from the differential form Eq. 25.

THE STREAM FUNCTION, VELOCITY AND RELATIVE ENTHALPY IN THE FRAME OF THE BLADE

Since ideal flow has been assumed in the moving frame of the blade, the stream function, ψ, is the usual function modified for motion along the y axis.

$$\psi = -W_{oy}(1 - a^2/r^2) r \cos\theta + W_{ox}(1 - a^2/r^2) r \sin\theta$$

$$+ (\Gamma/2\pi) \ln(r/a) \ . \tag{45}$$

The constant a is the radius of the cylinder. The cartesian velocity components are obtained by the usual transformation[18] as follows:

$$W_x = W_{ox} + \frac{a^2 W_{ox}(y^2 - x^2)}{(x^2 + y^2)^2} - \frac{2a^2 W_{oy} xy}{(x^2 + y^2)^2}$$

$$+ \frac{\Gamma}{2\pi} \frac{y}{(x^2 + y^2)} \ , \tag{46}$$

$$W_y = W_{oy} - \frac{2a^2 W_{ox} xy}{(x^2 + y^2)^2} + \frac{a^2 W_{oy}(x^2 - y^2)}{(x^2 + y^2)^2}$$

$$- \frac{\Gamma}{2\pi} \frac{x}{(x^2 + y^2)} \ . \tag{47}$$

Now the relative vorticity must vanish because potential flow cannot have vorticity. A check of the vorticity in the relative frame shows that indeed it vanishes. Also, the time-dependent term vanishes.

THE SUBSTANTIAL TOTAL ENTHALPY DERIVATIVE WITHOUT ROTATION

In the linear two-dimensional system, the differential form of the turbomachinery equation (Eq. 25) is simplified because the rotation vanishes.

$$\frac{Dh_o}{Dt} = \overline{U} \cdot (\overline{W} \cdot \nabla) \overline{W} \ . \tag{48}$$

Since the vorticity vanishes

$$\overline{W} \cdot \nabla \overline{W} = \nabla W^2/2 \ , \tag{49}$$

and employing Eq. 4

$$\frac{Dh_o}{Dt} = \overline{U} \cdot \nabla W^2/2 = \frac{U}{2} \frac{\partial W^2}{\partial y_w} \ . \tag{50}$$

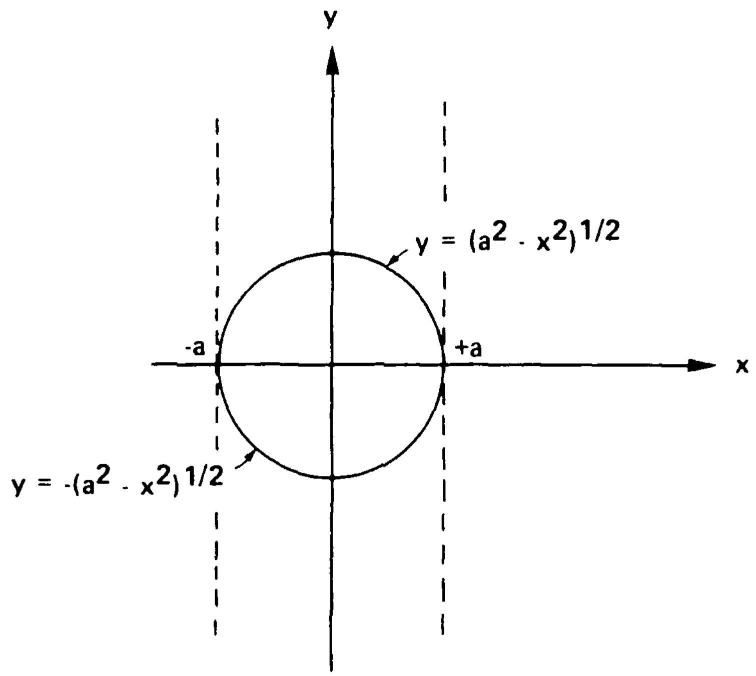


Fig. 4. Zone of interaction between the fluid and the blade.

The integrated substantial total enthalpy rate per unit length (where the subscript on y in Eq. 50 has been dropped) is

$$\frac{DH_o}{Dt} = \int\int \frac{\varrho U}{2} \frac{\partial(W_x^2 + W_y^2)}{\partial y} dydx \ . \tag{51}$$

Integration of Eq. 51 will be performed over all space per unit length z of the blade. The choice of time is immaterial since the fluid dynamics are steady state in the moving frame and the thermodynamic rates over all space are invariant with time. It is understood that the integration applies only to the fluid domain and that boundaries at solid walls are observed.

$$\frac{DH_o}{Dt} = \varrho U \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial W^2/2}{\partial y} dydx$$

$$= \varrho U \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d(W^2/2)dx \ . \tag{52}$$

See Fig. 4 for a definition of the boundary conditions of y on the blade wall. Since the kinetic energy in a conservative system, like thermodynamic quantities, is a function of state, the integration in Eq. 52 is a function of the end points only, which is verified by the fact that the integrand is a total differential.

$$\frac{DH_o}{Dt} = \varrho U \int_{-\infty}^{\infty} \left[\int_{-\infty}^{-(a^2-x^2)^{1/2}} d(W^2/2) + \int_{(a^2-x^2)^{1/2}}^{\infty} d(W^2/2) \right] dx \ . \tag{53}$$

Since the velocity is uniform at infinity (and all time and space derivatives vanish at infinity), the contributions at infinity will cancel in Eq. 53.

According to Fig. 4 integration of Eq. 53 with respect to x yields contributions from $-a \leq x \leq a$ only.

$$\frac{DH_o}{Dt} = -\varrho U \int_{-a}^{a} \int_{-(a^2-x^2)^{1/2}}^{(a^2-x^2)^{1/2}} d(W_x^2 + W_y^2)/2dx \ . \tag{54}$$

Now without going into details[18] the integration of the total derivative in Eq. 54 is performed, yielding

$$\begin{aligned}\frac{DH_o}{Dt} = -\varrho U \int_{-a}^{a} \Bigg(& \frac{4UW_{ox} \ x(a^2-x^2)^{1/2}(2x^2-a^2)}{a^4} \\ & + \frac{2W_{ox}\Gamma x(a^2-x^2)^{1/2}}{\pi a^4} + \frac{8W_{ox}Ux(a^2-x^2)^{1/2}}{a^2} \\ & + \frac{W_{ox}\Gamma(a^2-x^2)^{1/2}}{\pi a^4} + \frac{4W_{ox}Ux(a^2-x^2)^{1/2}(a^2-2x^2)}{a^4} \\ & + \frac{W_{ox}\Gamma(a^2-x^2)^{1/2}(a^2-2x^2)}{\pi a^4} \Bigg) dx \ . \end{aligned} \tag{55}$$

Note that only odd terms in y make any contribution to Eq. 55. Since the first and fifth terms cancel, only four terms remain. The integration with respect to x is performed through a transformation employing

$$x = a \cos\theta \ , \tag{56}$$

with integration limits given by

$$\begin{aligned} \theta = \pi \quad when \quad x = -a \ , \\ \theta = 0 \quad when \quad x = a \ . \end{aligned} \tag{57}$$

Making the substitutions

$$\frac{DH_o}{Dt} = -\varrho U\left(-\frac{2W_{ox}\Gamma}{\pi}\int_{\pi}^{0} \cos^2\theta \sin^2\theta d\theta \right.$$

$$- 8W_{ox}Ua \int_{\pi}^{0} \cos\theta \sin^2\theta d\theta - \frac{W_{ox}\Gamma}{\pi}\int_{\pi}^{0} \sin^2\theta d\theta$$

$$\left. -\frac{W_{ox}\Gamma}{\pi}\int_{\pi}^{0} \sin^2\theta(1-2\cos^2\theta)d\theta \right) . \tag{58}$$

In Eq. 58 the second integral makes no contribution because it is antisymmetric. The first integral cancels the second term in the last integral to yield from the surviving terms

$$\frac{DH_o}{Dt} = -\varrho W_{ox}\Gamma U = -\varrho V_{ox}\Gamma U . \tag{59}$$

Equation 59 is identical with Eq. 44 and this result illustrates a useful application of the differential form and constitutes confirmation of the validity of the differential turbomachinery Equation (Eq. 25). For the linear case, the energy transfer rate of the rotor is proportional to the component of the kinetic energy gradient parallel to the moving rotor (or sail).

Recall that energy transfer occurs in the narrow domain of integration indicated in Fig. 4 which ranges to infinity. Thus the velocity of sound must be infinite in the potential system in agreement with the assumption of incompressibility.

REFERENCES

1. Crocco, L., "Eine neue Stromfunction fur die Erforschung der Bewegung der Gase Mit Rotation." *Z. angew. Math. u. Mech.* Vol. 17, 1 (1937).

2. Tsien, H.S., *Fundamentals of Gas Dynamics*, Howard W. Emmons, Ed. Princeton University Press (1958).

3. Spannhake, W., "Die Leistungsaufnahme einer parallelkraenzigen Zentrifugalpumps mit radialen Schaufeln," *Z. angew. Math. u. Mech.*, p. 481 (1925).

4. Dean, R.C., "On the Necessity of Unsteady Flow in Fluid Mechanics," *ASME J. Basic Eng.*, pp. 24–28 (1959).

5. Greitzer, E.M., "An Introduction to Unsteady Flow in Turbomachines," published in *Thermodynamics and Fluid Mechanics of Turbomachinery*. Editors, Ucer, A.S., P. Stow, and Ch. Hirsch. Martinus Nijhoff Publishers, Dordrecht/Boston/Lancaster (1985).

6. Preston, J.H., "The Non-Steady Irrotational Flow of an Inviscid, Incompressible Fluid with Special Reference to Changes in Total Pressure through Flow Machines," *The Aeronautical Quarterly*, p. 343 (Nov 1961).

7. Lorenz, H., *Technische Hydrodynamik*, R. Oldenburg, Munich (1910).

8. Vavra, M.H., *Aero-Thermodynamics and Flow in Turbomachines*, John Wiley & Sons, Inc., New York, NY (1960).

9. Horlock, J.H., *Axial Flow Compressors*, Robert E. Krieger Publishing Co., Huntington, New York (1973).

10. Horlock, J.H. and H. Daneshyar, "Stagnation Pressure Changes in Unsteady Flow," *Aeronautical Quarterly*, Vol. XXII, Part III (Aug 1971).

11. Hawthorne, W.R., "The Flow Through Moving Cascades of Lifting Lines with Fluctuating Lift," Aeronautical Research Council, Paper No. ARC 32369 (1970).

12. Eck, B., *Technische Stromungslehre*, 5th ed. Berlin, Springer-Verlag p. 422 (1957).

13. Csanady,G.T., *Theory of Turbomachines*, New York, McGraw-Hill, p. 392 (1964).

14. Kemp, N.H. and W.R. Sears, Aerodynamic Interference Between Moving Blade Rows. *J. Aeronautical Sci.*, Vol. 20, No. 9, pp. 585–597 (Sep 1953).

15. Foa, J.V., *Elements of Flight Propulsion*, John Wiley & Sons, Inc., New York, pp. 83–84 (1960).

16. Wu Chung-Hua and Lincoln Wolfenstein, "Application of Radial-Equilibrium Condition to Axial Flow Compressor and Turbine Design," NACA Rept. 955, p. 165–194.

17. Wu Zhonghua, "Fundamental Aerothermodynamic Equations for Stationary and Moving Coordinate Systems," *Engineering Thermophysics in China*, Vol. 2 (1980).

18. Urbach, H.B., "Fluid Dynamics of Energy Transfer in Turbomachinery," Thesis, Department of Aerospace Engineering, University of Maryland (1987). See also internal report of the David Taylor Research Center, DTRC/TM–27–88–56 (Apr 1989).

INITIAL DISTRIBUTION

Copies

3 CONR
- 1 1132
- 1 1132F
- 1 121

4 NAVSEA
- 1 SEA 56X
- 1 SEA 56X7
- 1 SEA 55N

12 DTIC

1 Prof. D.G. Crighton
Deptartment of Applied Mathematical and Theoretical Physics
Cambridge Unirversity
Cambridge, England

1 Prof. E.M. Greitzer
Gas Turbine Laboratory
Massachusetts Institute of Technology
Cambridge, MA 02139

1 Dr. David Japiske
Concepts ETI, INC.
P.O. Box 643
Norwich, VT 05055

1 Prof G. Koopman
School of Engineering
Pennsylvania State University
University Park, PA 16802

1 Dr. D.B. Hanson
Hamilton Standard Division
United Technologies Corporation
Windsor Locks, CT

1 Dr. J. Hardin
NASA Langley Research Center
Hampton, VA 23665

1 Dr. R. Henderson
Applied Research Laboratory
P.O . Box 30
State College, PA 16804

1 Prof. A.K.M.F. Hussain
Department of Mechanical Engineering
University of Houston
Houston, TX 77004

1 Prof. James P. Johnston
Mechanical Engineering Dept.
Stanford University
Stanford, CA 94305–3030

1 Prof. D. McLaughlin
School of Engineering
Pennsylvanial State University
University Park, PA 16802

1 Dr. M. Rai
NASA Ames Research Center
Moffett Field, CA 94035

1 Prof. Donald Rockwell
Department of Mechanical Engineering
Lehigh University
Bethlehem, PA 18015

1 Dr. LeRoy Smith, Jr.
General Electric Company
Aircraft Engine Group H4
Cincinnati, OH 45215

1 Dr. D. Thompson
Applied Research Laboratory
P.O. Box 30
State College, PA 16801

1 Prof. T.Y. Wu
Mechanical Engineering Dept.
California Inst of Technology
Pasadena, CA 91125

CENTER DISTRIBUTION

Copies	Code	Name
1	0113	Dr. Bruce Douglas
1	15	
1	1508	
1	152	

1	154	
1	1544	
1	184	
1	19	
1	1905	
1	194	
1	1944	
1	27	
20	272T	
1	2721	
1	2721	
1	272R	
1	2722	
1	274	
1	2741	
1	3411	Knox
1	3431	